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Optimal Coordinates for Squire's Jet

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I. Introduction

THE role of the coordinate system in fluid mechanics problems has long intrigued workers in the field. Specifically, one often attempts to find coordinate systems in which a boundary-layer solution is uniformly valid throughout the entire flowfield. Such coordinates, following Kaplun,[1] are said to be "optimal." In his classic paper, which deals with steady, planar, incompressible flows, rules for constructing optimal coordinate systems are given.

For three-dimensional flows, the general problem concerning the existence of optimal coordinates is still unsolved. In this paper, the problem of the laminar round jet (without swirl), issuing into an infinite fluid at rest, is considered. The flow here is assumed to be steady and incompressible. An exact solution for the entire flowfield exists, and is discussed at length by Batchelor[2] and Yih.[3] The solution was first obtained by Landau in 1944, but his results were never published, and, independently, by Squire[4] in 1951. More recently, the same results were derived by Coles[5] using the method of matched asymptotic expansions.

In the above analyses, the governing equations were posed in spherical coordinates. Simple geometric considerations show that the outer (potential flow) stream surfaces are paraboloids. Thus, the choice of a paraboloidal coordinate system at least appears to be a natural one, and here, the method of inner and outer expansions is applied to the Navier-Stokes equations written in this system. It is shown that the solution obtained describes an optimal solution; more surprisingly, the optimal coordinate system obtained here coincides with the stream-surface coordinate system of the outer entrained flow. Also, as expected, the solution in paraboloidal coordinates takes on a particularly simplified form.

For the momentum equations written in spherical coordinates, Squire has shown that the stream function

$$\psi = 2\nu r \sin^2\theta / (\alpha + 1 - \cos\theta) \tag{1}$$

describes the full solution to the laminar round jet. It is noted that r measures the distance from the jet source; θ measures the "width" or "jet half-angle" (taking the source point as vertex); and, ϕ measures the angular position in planes perpendicular to the jet axis (i.e., swirl coordinate). Here, the velocity components in the $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ directions are, respectively,

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \text{ and } v_\phi = 0 \tag{2}$$

where ν denotes the kinematic viscosity, and α is a parameter related to the jet width. Following the development of Coles,[5] if, for nonzero α and small θ, the θ's in Eq. (1) are normalized by $\theta_0 \simeq \sqrt{2\alpha}$ (i.e., the angle for which the jet stream surfaces become parallel to the jet axis), denoting $\psi = \psi^i$, and $\xi = \theta/\theta_0$, it is easy to verify that $\lim_{\xi\to\infty} \psi^i = 4\nu r$.

Alternatively, with α identically zero, and denoting $\psi = \psi^0$, one observes also that $\lim_{\theta\to 0} \psi^0 = 4\nu r$. Clearly, from setting $\alpha = 0$ in Eq. (1), the potential flow is given by $\psi^0 = 2\nu r(1+\cos\theta)$. From this, it is readily seen that surfaces of $\psi^0 =$ constant are paraboloidal. With the inner solution $\psi^i = 4\nu r\xi^2/(1+\xi^2)$, one may easily construct a composite solution $\psi^c = \psi^0 + \psi^i - 4\nu r$. It has also been shown[4] that the momentum flux M across the surface S of any sphere centered at the origin is

$$M = 2\pi\rho\nu^2 \left\{ \frac{32(\alpha+1)}{3\alpha(\alpha+2)} - 16(\alpha+1) + 8\alpha(\alpha+2)\log\frac{\alpha+2}{\alpha} \right\} \tag{3}$$

while the force in the axial direction exerted by the sphere on the surrounding fluid is

$$F = 2\pi\rho\nu^2 \left\{ 24(\alpha+1) - (12\alpha^2 + 24\alpha + 4)\log\frac{\alpha+2}{\alpha} \right\} \tag{4}$$

The sum $M + F$, of course, can be considered as the strength of a momentum source situated at the origin.

II. Solution in Paraboloidal Coordinates

When (u,v,ϕ) denote the curvilinear variables in the paraboloidal coordinate system (Fig. 1), the full equations governing the problem are:

$$\begin{aligned} A(A_u + BvR) + B(A_v - BuR) &= -\frac{P_u}{\rho} \\ &+ \frac{\nu R}{uv}\{(2uvA_u + 2uv^2BR)_u - 2B - 2Av/u \\ &+ (uvA_v + uvB_u - uv^2AR - u^2vBR)_v \\ &+ uv^2R(A_v + B_u - vAR - uBR) + 2u^2vR(-B_v - uAR)\} \end{aligned} \tag{5a}$$

$$\begin{aligned} A(B_u - AvR) &+ B(B_v + AuR) \\ &= -\frac{P_v}{\rho} + \frac{\nu R}{uv}\{(2uvB_v + 2u^2vAR)_v - 2A - 2Bu/v \\ &+ (uvA_v + uvB_u - u^2vBR - uv^2AR)_u \\ &+ u^2vR(A_v + B_u - vAR - uBR) + 2uv^2R(-A_u - vBR)\} \end{aligned} \tag{5b}$$

and

$$(uvA/R)_u + (uvB/R)_v = 0 \tag{5c}$$

In the above, the velocity vector was taken in the form $\hat{q} = A\hat{e}_u + B\hat{e}_v + C\hat{e}_\phi$. For convenience, partial derivatives are indicated by subscripts, and $u^2 + v^2 = 1/R$. By invoking the condition of no swirl, it is evident that $C = 0$, and further, that partial derivatives with respect to ϕ vanish. Equations (5a) and (5b) describe the momentum balance in the $\hat{e}_u$ and $\hat{e}_v$ directions, respectively; Eq. (5c) specifies the condition of mass continuity. These relations need to be supplemented by the conditions of zero flow far from the jet axis, of symmetry, and of some prescribed momentum flux.

For the inner region occupied by the jet, it is appropriate to introduce changes in scale defined by the new variables, $u^* = u$, $v^* = v/\epsilon$, $A = A^*(u^*,v^*) + O(\epsilon)$, $B = \epsilon B^*(u^*, v^*) + O(\epsilon^2)$, and $P = P^*(u^*, v^*) + O(\epsilon)$. Equations (5a-c) may be written in nondimensional form (in terms of a Reynolds number based on, say, the centerline velocity); the prior

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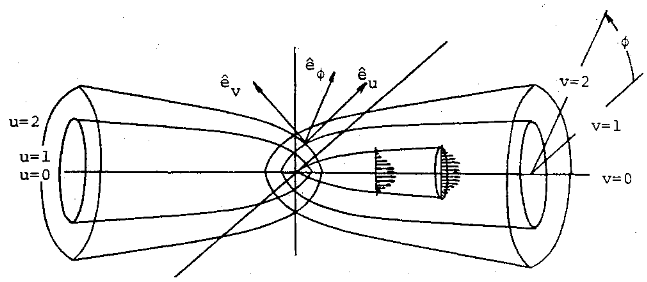

Fig. 1 Round jet in paraboloidal coordinates.

substitutions reproduce the results of classical boundary-layer theory, i.e., $\epsilon = 1/\sqrt{Rey}$. Thus, it is seen that to first order, taking $P^* \equiv P_\infty$, the governing equations reduce to, returning to dimensional quantities,

$$AA_u + BA_v = \frac{\nu}{u^2 v}(uvA_v)_v \tag{6a}$$

$$(u^2 vA)_u + (u^2 vB)_v = 0 \tag{6b}$$

The "modified" continuity equation (6b) now may be integrated by the introduction of a stream function Ψ^i defined by $u^2vA = \Psi^i_v$ and $u^2vB = -\Psi^i_u$ where subscripts again denote partial differentiation. The momentum equation (6a) is thereby transformed to

$$\Psi^i_v \frac{\partial}{\partial u}\left(\frac{1}{u^2 v}\frac{\partial \Psi^i}{\partial v}\right) - \Psi^i_u \frac{\partial}{\partial v}\left(\frac{1}{u^2 v}\frac{\partial \Psi^i}{\partial v}\right) = \nu\frac{\partial}{\partial v}\left(uv\frac{\partial}{\partial v}\frac{1}{u^2 v}\frac{\partial \Psi^i}{\partial v}\right) \tag{7}$$

It is apparent from dimensional considerations that the integration of Eq. (7) may be more easily accomplished by taking $\eta = v/u$ as an independent variable and by taking Ψ^i in the form $\Psi^i = \nu u^2 f(\eta)$. With this assumption, the conditions of symmetry and finite velocity along the jet axis requires $f(0) = 0$. The condition of zero flow far from the jet axis requires that $f'(\infty) = f''(\infty) = 0$. The latter boundary conditions (at infinity) are not independent of each other; still another condition needs to be prescribed. Physically, this condition is related to the strength of the jet; the constant associated with this integration is analagous to the α of Sec. I.

Equation (7) may now be cast into the more simplified form

$$f''' - \frac{1}{\eta}f'' + \frac{1}{\eta^2}f' = \frac{2}{\eta^2}ff' - \frac{2}{\eta}ff'' - \frac{2}{\eta}(f')^2 \tag{8}$$

or, alternatively,

$$(f'')' - (f'/\eta)' = -2(ff'/\eta)'$$

Here, the primes indicate differentiation with respect to η. Successive integrations yield

$$(2ff' + \eta f'' - f')/\eta = \gamma \qquad \text{with } \gamma = 0$$

$$\eta f' + f^2 - 2f = \beta \qquad \text{with } \beta = 0$$

and, finally,

$$f = 2\eta^2/(\eta^2 + \kappa)$$

Thus, the first-order boundary-layer solution is given by

$$\Psi^i = 2\nu u^2 \eta^2/(\eta^2 + \kappa) \tag{9}$$

where κ is related to the momentum strength (or jet width). I is an easy matter to establish the relation $\alpha = 2\kappa$ from simpl coordinate transformations; then, Eqs. (3) and (4) are ex pected to remain valid so that the flow is completely deter mined when M, F, or $M+F$ are specified.

For the outer flow, the independent variables of Eqs. (5a-c do not require stretching; consequently, to first order, on recovers the inviscid Euler equations. The solution to thi order, because the flow is irrotational far upstream, satisfie $\nabla \times \hat{q} = 0$. This condition, together with a definition of stream function Ψ^0 appropriate to Eq. (5c), i.e., $uvA/R = \Psi$ and $uvB/R = -\Psi^0_u$, determines the governing potential flo equation

$$\Psi^0_{uu} + \Psi^0_{vv} - \Psi^0_u/u - \Psi^0_v/v = 0 \tag{1C}$$

The corresponding ("line sink") boundary condition on th jet axis is obtained from the limit matching principle. Henc it is required that $\lim_{v\to 0} \Psi^0 = \lim_{\eta\to\infty} \Psi^i$, i.e., $\Psi^0(v=0) = 2\nu u^2$ In addition, the solution to Eq. (10) must satisfy the conditio of zero flow far from the jet axis. The outer (potential) flo solution in this case is obviously given by the inner boundar condition; that is, $\Psi^0(u,v) = 2\nu u^2$. (This, of cours reproduces the fact that the outer stream surfaces ar paraboloidal.)

III. Conclusion

It is seen from the results of Sec. II that the composi solution $\Psi^c = \Psi^i + \Psi^0 - 2\nu u^2$ is identical to Ψ^i. Thus, for th laminar round jet problem, an optimal coordinate system h been discovered. It has been shown by Kaplun[1] that in th case of planar flows, optimal coordinates are not unique. I the general case, such coordinates are not expected to coincid with streamline coordinates. The results of this paper sugge that, for Squire's jet, the connection between the two types c coordinate systems is stronger. For further discussion o optimal coordinates, the reader is referred to Ref. 6.

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A Study of Multiple Jets

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Nomenclature

D_0 = diameter of a single jet, 3.57 mm
f = octave band center frequency
m = jet momentum
R = radial distance from the jet(s) central axis
$R_{1/2}$ = radial distance at which the velocity is half t central velocity
S = Strouhal number, fD_0/V_j

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